लक्ष्यं विश्वमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2023

MTMACOR13T-MATHEMATICS (CC13)

Time Allotted: 2 Hours　　　　　　　　　　　　　　　　　　　　Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any *five* questions from the following: $2\times5=10$

(a) Let (X,d) be a metric space and $A\subset X$. Then show that $\overline{A}$ is a closed set where $\overline{A}$ is the closure of A. 2

(b) Let $X=(0,1]$ be the metric space with usual metric and $\{x_n\}_n$ where $x_n=\frac{1}{n}$ be a sequence in X. Show that $\{x_n\}_n$ is a Cauchy sequence. Is X complete? Justify your answer. 2

(c) Give an example of a complete metric space and an incomplete metric space. 2

(d) Show that in a metric space any two disjoint sets are always separated. 2

(e) Evaluate $\int_0^{2+i}\overline{z}^2\,dz$ along the line $2y=x$. 2

(f) Find $\lim\limits_{z\to i}\dfrac{\overline{z}+z^2}{1-\overline{z}}$. 2

(g) Show that the function $f(z)=|z|^2$ is continuous everywhere on C. 2

(h) Prove that $f(z)=\operatorname{Re} z$ is nowhere differentiable. 2

2. (a) Let (X,d) be a metric space. Then show that $\sigma(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$ is a bounded metric on X, $x,y\in X$. 5

(b) Prove that every Cauchy sequence in a metric space is bounded. 3

3. (a) State and prove Cantor's intersection theorem. 4

(b) Let $\{x_n\}$, $\{y_n\}$ be two convergent sequences in a metric space (X,d) and converge to $x,y\in X$ respectively then prove that the sequence $\{d(x_n,y_n)\}$ of real numbers converges to $d(x,y)$ in usual real metric space. 4

4. (a) Let (X, d_1) and (Y, d_2) be two metric spaces and $f: X \to Y$ be a function. Then show that f is continuous at a point $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}_n \subset X$ with $x_n \to x_0$. 4

(b) Let X, Y be two metric spaces. If $f: X \to Y$ is continuous then show that for every compact subset $E \subset X$, the image $f(E)$ is a compact subset of Y. 4

5. (a) Let (X, d) be a metric space, $G \subset X$ and $G = A \cup B$ where A and B are separated sets. Show that if G is open then A and B are open. 4

(b) Prove that $A \subset \mathbb{R}$ is connected with respect to usual metric if and only if it is an interval. 4

6. (a) Let $f: D \longrightarrow \mathbb{C}$ $(D \subset \mathbb{C})$ be a complex valued function and $z_0 \in D$, where $f(z) = u(x, y) + i\, v(x, y)$ $(x, y \in \mathbb{R})$. Then show that the function $f(z)$ is continuous at $z_0 = x_0 + iy_0$ iff $u(x, y)$ and $v(x, y)$ are continuous at $(x_0; y_0)$. 4

(b) If the complex sequence $\{z_n\}_n$ where $z_n = a_n + ib_n$, $n \in \mathbb{N}$ converges to $z = a + ib$, then prove that $\{|z_n|\}_n$ converges to $|z|$. Is the converse of the above result true? Justify your answer. 2+2

7. (a) Let $f(z) = u(x, y) + i\, v(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D and first order partial derivative of u and v satisfy Cauchy-Riemann equations at a point $(x, y) \in D$, then prove that f is differentiable at $z = x + iy$. 4

(b) Show that the function $f(z) = u + iv$ where 4

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \qquad z \neq 0$$
$$= 0 \qquad z = 0$$

is not differentiable at the origin even though it satisfy Cauchy-Riemann equations at the origin.

8. (a) State and prove Cauchy's integral formula for the first order derivative of an analytic function. 5

(b) Show that $\oint\limits_{\gamma} \frac{z+4}{z^2 + 2z + 5}\, dz = 0$. 3

9. (a) If f is an integral function and $|f(z)| < M$, for all z, M being a positive constant then prove that f is constant. 4

(b) Find Taylor series expansion of $f(z) = \frac{z-1}{(z+1)}$ 4

(i) about the point $z = 0$
(ii) about the point $z = 1$.

Determine the region of convergence in each case.

——×——